

Parametric amplification of magnetoplasmons in semiconductor quantum dots

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We show that the magnetoplasmon collective modes in quasi-two-dimensional semiconductor quantum dots can be parametrically amplified by periodically modulating the magnetic field perpendicular to the nanostructure. The two magnetoplasmon modes are excited and amplified simultaneously, leading to an exponential growth of the number of bosonic excitations in the system. We further demonstrate that damping mechanisms as well as anharmonicities in the confinement of the quantum dot lead to a saturation of the parametric amplification. This work constitutes a first step towards parametric amplification of collective modes in many-body fermionic systems beyond one dimension.

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I. INTRODUCTION

The swing is the paradigm for the phenomenon of parametric resonance:¹ The periodic motion of the swinger's legs leads to a periodic modulation of the effective length of the pendulum, hence of its frequency. This results in the exponential amplification of the motion if the period of the modulation is chosen to be commensurate with the natural frequency of the pendulum. Quantum mechanically, if one considers a single harmonic oscillator whose frequency is periodically varied in time, the parametric modulation of the Hamiltonian leads to an exponential divergence of the corresponding lowering and raising operators, hence increasing the number of bosonic quanta in the system. As in the classical case,¹ this effect is most efficient if the pumping frequency is twice as much as the natural frequency of the oscillator. For a parametrically modulated many-body bosonic system, only the eigenmodes fulfilling the resonance condition are amplified.^{2,3} Parametric resonance is thus an interesting and valuable spectroscopic tool for many-body quantum systems, especially in the context of cold atomic gases⁴ where the tunability of the experimental setups enables one to parametrically amplify Bose-Einstein condensates.^{5,6}

In this paper, we address the question of parametric resonances in many-body *fermionic* systems. In contrast to the bosonic case, any mean-field treatment of the interactions between fermions preserves the fermionic nature of the quasiparticles: The parametric amplification of fermionic modes is thus blocked due to the Pauli principle. However, many-body fermionic systems are known to exhibit bosonic collective modes such as, e.g., plasmons (charge density waves) and magnons (spin waves).^{7,8} It has been shown in various contexts that such collective modes can be parametrically excited in one-dimensional correlated fermionic systems.^{9–12} Using the Luttinger liquid picture and bosonization techniques which enable one to treat the interactions among the particles (almost) exactly,^{8,13} it was proposed that the parametric modulation of the optical lattice where cold

fermionic atoms are trapped could serve as a probe for the well-known spin-charge separation inherent to one-dimensional correlated fermionic systems.^{10,11}

Extending these ideas to two- and three-dimensional many-body fermionic systems is a formidable theoretical challenge, as one needs to capture the relevant correlations by treating interactions beyond mean field. In this paper, we focus on the case of *confined* many-body fermionic systems, where collective modes exist due to the finite size of the system.¹⁴ Examples of such collective modes are plasmon excitations which correspond to the motion of the electronic center of mass in metallic nanoparticles^{14–16} and in quasi-two-dimensional semiconductor quantum dots.^{17–19}

It is well known that a static magnetic field induces field-dependent collective modes termed magnetoplasmons.^{17,18} These modes appear in particular in quasi-two-dimensional semiconductor quantum dots,^{20–22} and were recently investigated theoretically in metallic nanoparticles.¹⁶ The magnetoplasmons, whose classical motion corresponds to an excitation of the electronic center of mass perpendicular to the magnetic field rotating clockwise and counterclockwise (see the inset in Fig. 1), have distinct frequencies that depend on the applied static magnetic field (see Fig. 1). In this work, we propose to trigger the parametric amplification of the magnetoplasmons by periodically modulating the magnetic field perpendicular to the nanostructure. Specifically, we focus on quasi-two-dimensional semiconductor quantum dots. We show that the parametric modulation of the magnetic field leads to an exponential growth of the number of bosonic modes in the system. We further demonstrate how damping mechanisms and anharmonicities in the confinement of the quantum dot lead to the saturation of the parametric amplification.

This article is organized as follows: We start in Sec. II by presenting our model of a two-dimensional quantum dot subject to a perpendicular magnetic field which is periodically modulated in time. In Sec. III, we show that such a modulation leads to the parametric ampli-

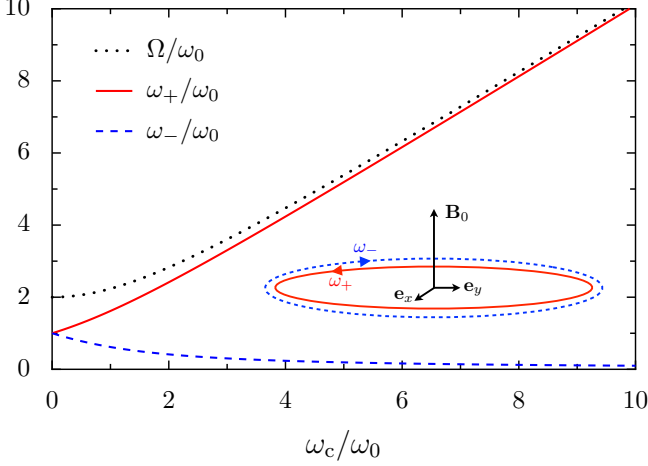


FIG. 1. (Color online) Frequencies ω_+ and ω_- [cf. Eq. (14)] of the two magnetoplasmon modes (rescaled by the confinement frequency ω_0) as a function of the cyclotron frequency $\omega_c = eB_0/m$. The pumping frequency exactly at resonance, $\Omega = \omega_+ + \omega_-$, is also shown as a dotted line. Inset: Sketch of the two independent center-of-mass motions associated to the surface magnetoplasmon modes perpendicular to the static magnetic field \mathbf{B}_0 .

fication of the magnetoplasmon collective modes. The effect of damping and anharmonicities on the parametric resonance of the magnetoplasmons is studied in detail in Sec. IV. We discuss in Sec. V the experimental consequences of our proposal before we conclude in Sec. VI. Several technical aspects of our work are presented in the appendices.

II. MODEL

We consider N_e interacting electrons with effective mass m^* and charge $-e$ confined in a two-dimensional quantum dot. The quantum dot is subject to a spatially homogeneous perpendicular magnetic field $\mathbf{B}(t) = B(t)\mathbf{e}_z$ with

$$B(t) = B_0[1 + \eta \sin(\Omega t)] \quad (1)$$

which is periodically modulated at the frequency Ω . In what follows, we assume the strength of the periodic modulation η to be much smaller than one. The Hamiltonian of the system reads

$$H(t) = \sum_{i=1}^{N_e} \left[\frac{1}{2m^*} (\mathbf{p}_i + e\mathbf{A}(\mathbf{r}_i, t))^2 + U(r_i) \right] + \sum_{\substack{i,j=1 \\ (i \neq j)}}^{N_e} V_{e-e}(|\mathbf{r}_i - \mathbf{r}_j|), \quad (2)$$

with $\mathbf{r}_i = x_i\mathbf{e}_x + y_i\mathbf{e}_y$ the position of the i th electron, \mathbf{p}_i its momentum and $\mathbf{A}(\mathbf{r}, t)$ the time-dependent vector

potential. Notice that we do not consider the spin degree of freedom as it is not relevant for understanding the parametric amplification of the magnetoplasmons, as the latter only involves the orbital degrees of freedom. Hence, we neglect the Zeeman term (which, for confined semiconductor quantum dots is relatively weak due to the small value of the associated g factor) and the spin-orbit coupling in Eq. (2). The single-particle confinement $U(r)$ appearing in Eq. (2) is approximated by a parabolic potential with confining frequency ω_0 .^{17,18} Including also quartic corrections that will be relevant for the following analysis (see in particular Sec. IV), we write it as

$$U(r) = \frac{m^*\omega_0^2}{2}r^2 + \frac{a_4}{4}r^4, \quad (3)$$

with $a_4 > 0$. Finally, V_{e-e} in Eq. (2) stands for the electron-electron interaction. In the symmetric gauge

$$\mathbf{A}(\mathbf{r}, t) = \frac{B(t)}{2}(-y\mathbf{e}_x + x\mathbf{e}_y), \quad (4)$$

one can rewrite the Hamiltonian (2) as

$$H(t) = \sum_i \left[\frac{p_i^2}{2m^*} + U(r_i) + \frac{\omega(t)}{2}l_{z,i} + \frac{m^*\omega^2(t)}{8}r_i^2 \right] + \sum_{ij} V_{e-e}(|\mathbf{r}_i - \mathbf{r}_j|), \quad (5)$$

with $l_{z,i} = (\mathbf{r}_i \times \mathbf{p}_i)_z$ the z -component of the angular momentum, $\omega(t) = eB(t)/m^* = \omega_c[1 + \eta \sin(\Omega t)]$, and $\omega_c = eB_0/m^*$ the cyclotron frequency. As we will show in Sec. III, the diamagnetic term $\propto \omega^2(t)$ in Eq. (5) is responsible for the parametric amplification of the magnetoplasmon collective modes.

Introducing the collective coordinate for the electronic center of mass $\mathbf{R} = X\mathbf{e}_x + Y\mathbf{e}_y = \sum_i \mathbf{r}_i/N$ and its conjugated momentum $\mathbf{P} = P_X\mathbf{e}_x + P_Y\mathbf{e}_y = \sum_i \mathbf{p}_i$, as well as the relative degrees of freedom $\mathbf{r}'_i = \mathbf{r}_i - \mathbf{R}$ and $\mathbf{p}'_i = \mathbf{p}_i - \mathbf{P}/N$,^{15,16} the Hamiltonian (5) separates into

$$H(t) = H_{\text{cm}}(t) + H_{\text{rel}}(t) + H_c, \quad (6)$$

with the center-of-mass Hamiltonian

$$H_{\text{cm}}(t) = \frac{P^2}{2M} + \frac{M}{2} \left(\omega_0^2 + \frac{\omega^2(t)}{4} \right) R^2 + \frac{\omega(t)}{2} L_Z + \frac{A_4}{4} R^4, \quad (7)$$

where $M = N_e m^*$, $A_4 = N_e a_4$, and $L_Z = X P_Y - Y P_X$. The Hamiltonian for the relative coordinates reads

$$H_{\text{rel}}(t) = \sum_i \left[\frac{p_i'^2}{2m^*} + U(r'_i) + \frac{\omega(t)}{2} l'_{z,i} + \frac{m^*\omega^2(t)}{8} r_i'^2 \right] + \sum_{ij} V_{e-e}(|\mathbf{r}'_i - \mathbf{r}'_j|). \quad (8)$$

Finally, the quartic part of the single-particle confinement (3) leads to the coupling between the center of mass and the relative coordinates in Eq. (6), with Hamiltonian

$$H_c = \frac{a_4}{4} \sum_i \left(|\mathbf{R} + \mathbf{r}'_i|^4 - R^4 - r_i'^4 \right). \quad (9)$$

Notice that if one assumes the single-particle confinement (3) to be purely harmonic ($a_4 = 0$), there is no coupling in Eq. (6) between the electronic center of mass and the relative coordinates, as requested by the generalized Kohn's theorem.^{17,18,23,24} The coupling (9) induces the damping of the center-of-mass excitations, and leads to a finite linewidth of the magnetoplasmon lines in far infrared absorption spectroscopy experiments.^{17,18}

The time-independent, quadratic part of the center-of-mass Hamiltonian (7) can be diagonalized by means of Fock-Darwin states in terms of two magnetoplasmon excitations.¹⁷ Introducing the variable $\xi = (X + iY)/\sqrt{2}$ and the bosonic operators associated to the magnetoplasmons

$$b_+ = \frac{1}{\sqrt{2}} \left(\frac{\xi^*}{\ell} + \ell \frac{\partial}{\partial \xi} \right), \quad b_+^\dagger = \frac{1}{\sqrt{2}} \left(\frac{\xi}{\ell} - \ell \frac{\partial}{\partial \xi^*} \right), \quad (10a)$$

$$b_- = \frac{1}{\sqrt{2}} \left(\frac{\xi}{\ell} + \ell \frac{\partial}{\partial \xi^*} \right), \quad b_-^\dagger = \frac{1}{\sqrt{2}} \left(\frac{\xi^*}{\ell} - \ell \frac{\partial}{\partial \xi} \right), \quad (10b)$$

with the oscillator length

$$\ell = \sqrt{\frac{\hbar}{M(\omega_0^2 + \omega_c^2/4)^{1/2}}}, \quad (11)$$

we rewrite Eq. (7) as

$$\begin{aligned} H_{\text{cm}}(t) = & \sum_{\sigma=\pm} \hbar \omega_\sigma [1 + \sigma \delta f(t)] \left(b_\sigma^\dagger b_\sigma + \frac{1}{2} \right) \\ & + \alpha \frac{\hbar \omega_0}{4} (b_+^\dagger + b_-) (b_+ + b_-^\dagger) \\ & + \frac{\hbar \omega_c}{2} \delta f(t) (b_+^\dagger b_-^\dagger + b_- b_+). \end{aligned} \quad (12)$$

In Eq. (12),

$$\delta f(t) = \eta \frac{\omega_c/2}{\sqrt{\omega_0^2 + \omega_c^2/4}} \sin(\Omega t), \quad (13)$$

and we introduced the dimensionless parameter $\alpha = A_4 \ell^4 / \hbar \omega_0 \ll 1$. The frequencies of the two magnetoplasmon collective modes read¹⁷

$$\omega_\pm = \sqrt{\omega_0^2 + \frac{\omega_c^2}{4}} \pm \frac{\omega_c}{2} \quad (14)$$

and are presented in Fig. 1 as a function of the cyclotron frequency ω_c . Without parametric modulation ($\eta = 0$), the center-of-mass motion associated with the magnetoplasmon with frequency ω_+ (ω_-) rotates counterclockwise (clockwise) in the plane perpendicular to the magnetic field (see the inset in Fig. 1). Notice that one has $\omega_+ > \omega_-$ due to the fact that the Lorentz force $-N_e e \dot{\mathbf{R}} \times \mathbf{B}_0$ increases (decreases) the effective harmonic confinement seen by the collective mode with frequency ω_+ (ω_-).

The last symmetry breaking term $\propto b_+^\dagger b_-^\dagger + b_+ b_-$ in the Hamiltonian (12) is triggered by the parametric modulation of the magnetic field [see Eq. (13)]. As we will show in Sec. III, this term is responsible for the parametric amplification of the two magnetoplasmon collective modes, provided that the pumping frequency Ω is close to the resonance condition $\Omega = \omega_+ + \omega_-$ (see dotted line in Fig. 1). The quartic corrections proportional to the parameter α in the Hamiltonian (12) represent a residual interaction between the two bosonic modes, and lead to the saturation of the parametric amplification (see Sec. IV).

In the absence of quartic corrections ($\alpha = 0$), the Hamiltonian (12) is similar to the one encountered in the quantum theory of parametric amplifiers in quantum optics.^{25–29} In this context, a pump laser interacts with a nonlinear crystal. Due to the second-order susceptibility of the nonlinear media, a photon of the pump laser at frequency Ω splits into two photons at frequencies ω_+ and ω_- , a phenomenon called parametric down conversion. In a parametric amplifier, the two signals at frequencies ω_+ and ω_- are amplified by pumping the crystal at $\Omega = \omega_+ + \omega_-$.

III. PARAMETRIC RESONANCE OF THE MAGNETOPLASMONS

We start our study of the possibility of parametrically amplifying the magnetoplasmon collective modes by neglecting the quartic part of the center-of-mass Hamiltonian, setting $\alpha = 0$ in Eq. (12). Furthermore, we neglect all other sources of dissipation on top of the one induced by the quartic confinement. (For the simpler case of a single harmonic oscillator with periodically-modulated frequency, see Appendix A.) The Heisenberg equations of motion for the bosonic operators appearing in the center-of-mass Hamiltonian (12) thus read

$$\dot{b}_\pm = -i\omega_\pm [1 \pm \delta f(t)] b_\pm - i\frac{\omega_c}{2} \delta f(t) b_\mp^\dagger. \quad (15)$$

Introducing

$$b_\pm(t) = \exp \left(-i\omega_\pm \int_0^t ds [1 \pm \delta f(s)] \right) \tilde{b}_\pm(t), \quad (16)$$

Eq. (15) transforms, to first order in the small parameter η , into

$$\dot{\tilde{b}}_\pm = -i\frac{\omega_c}{2} \delta f(t) e^{i(\omega_+ + \omega_-)t} \tilde{b}_\mp^\dagger. \quad (17)$$

The main contribution to the time evolution of these operators, within the rotating wave approximation, comes from the terms close to resonance, i.e., $\Omega \simeq \omega_+ + \omega_-$ [cf. Eq. (13)], yielding

$$\dot{\tilde{b}}_\pm \simeq \frac{\epsilon}{2} e^{-i(\Omega - \omega_+ - \omega_-)t} \tilde{b}_\mp^\dagger, \quad (18)$$

where we introduced the “driving” frequency

$$\epsilon = \eta \frac{(\omega_c/2)^2}{\sqrt{\omega_0^2 + \omega_c^2/4}}. \quad (19)$$

Equation (18) can be easily decoupled, yielding

$$\ddot{\tilde{b}}_{\pm} + i(\Omega - \omega_+ - \omega_-) \dot{\tilde{b}}_{\pm} - \frac{\epsilon^2}{4} \tilde{b}_{\pm} = 0. \quad (20)$$

Using the initial conditions $\tilde{b}_{\pm}(0) = b_{\pm}(0)$ and $\dot{\tilde{b}}_{\pm}(0) = \epsilon b_{\pm}^{\dagger}(0)/2$ [cf. Eq. (18)], we finally obtain the solutions

$$b_{\pm}(t) = \frac{e^{-i\omega_{\pm}t}}{\Omega_- - \Omega_+} \left[(\Omega_- e^{i\Omega_+t} - \Omega_+ e^{i\Omega_-t}) b_{\pm}(0) + i \frac{\epsilon}{2} (e^{i\Omega_+t} - e^{i\Omega_-t}) b_{\mp}^{\dagger}(0) \right], \quad (21)$$

with

$$\Omega_{\pm} = \frac{\omega_+ + \omega_- - \Omega}{2} \pm \frac{1}{2} \sqrt{(\Omega - \omega_+ - \omega_-)^2 - \epsilon^2}. \quad (22)$$

We thus see that tuning the pumping frequency Ω in the interval $|\Omega - \omega_+ - \omega_-| < \epsilon$, the frequencies Ω_{\pm} acquire an imaginary part, leading to the parametric amplification of the bosonic magnetoplasmon collective modes. (For a classical treatment of the parametric resonance of the magnetoplasmons and specifically of the associated classical trajectories of the electronic center of mass, see Appendix B.) Indeed, close to resonance, i.e., for $|\Omega - \omega_+ - \omega_-| \ll \epsilon$, Eq. (21) simplifies to

$$b_{\pm}(t) = e^{-i\omega_{\pm}t} \left[\cosh\left(\frac{\epsilon t}{2}\right) b_{\pm}(0) + \sinh\left(\frac{\epsilon t}{2}\right) b_{\mp}^{\dagger}(0) \right], \quad (23)$$

showing that the two bosonic modes are exponentially amplified at a rate given by the frequency ϵ of Eq. (19). On the contrary, off resonance ($|\Omega - \omega_+ - \omega_-| \gg \epsilon$), the bosonic operators evolve in time according to their unitary evolution,

$$b_{\pm}(t) = e^{-i\omega_{\pm}t} b_{\pm}(0). \quad (24)$$

Notice that the exponential amplification of the operators (21) is a direct consequence of their bosonic nature. If the operators b_{\pm} were fermionic, the Pauli principle would prevent the parametric amplification from occurring (see Appendix C for details).

The center-of-mass subsystem is, prior to the parametric modulation that starts at $t = 0$, in thermal equilibrium at the temperature T and thus described by the density matrix $\rho_0 = e^{-H_{\text{cm}}(0)/k_B T} / \mathcal{Z}$ with $\mathcal{Z} = \text{tr}\{e^{-H_{\text{cm}}(0)/k_B T}\}$ the canonical partition function. Introducing the average number of bosons in each mode

$$N_{\pm}(t) = \langle b_{\pm}^{\dagger} b_{\pm} \rangle(t), \quad (25)$$

as well as the “anomalous” average

$$\varphi(t) = \langle b_+ b_- \rangle(t) \quad (26)$$

where $\langle \mathcal{O} \rangle(t) = \text{tr}\{\rho_0 \mathcal{O}(t)\}$, we obtain using Eq. (23)

$$N_{\pm}(t) = \frac{1}{2} \{ [1 + n_B(\omega_+) + n_B(\omega_-)] \cosh(\epsilon t) + n_B(\omega_{\pm}) - 1 - n_B(\omega_{\mp}) \} \quad (27)$$

and

$$\varphi(t) = \frac{1}{2} e^{-i\Omega t} [1 + n_B(\omega_+) + n_B(\omega_-)] \sinh(\epsilon t) \quad (28)$$

when the pumping frequency is close to resonance. In Eqs. (27) and (28), we introduced the Bose distribution $n_B(\omega) = [\exp(\hbar\omega/k_B T) - 1]^{-1}$. Due to the fact that the magnetoplasmons are purely harmonic excitations of the electronic center of mass, $\langle R \rangle(t) = 0$, the fluctuations of the center-of-mass coordinate $\langle \delta R^2 \rangle(t) = \langle R^2 \rangle(t) - \langle R \rangle^2(t)$ reduce to

$$\langle \delta R^2 \rangle(t) = \ell^2 [N_+(t) + N_-(t) + 1 + \varphi(t) + \varphi^*(t)]. \quad (29)$$

With Eqs. (27) and (28), we obtain

$$\langle \delta R^2 \rangle(t) = \ell^2 [1 + n_B(\omega_+) + n_B(\omega_-)] \times [\cosh(\epsilon t) + \cos(\Omega t) \sinh(\epsilon t)]. \quad (30)$$

Since $\epsilon \ll \Omega$, one can average this equation over a timescale long compared to Ω^{-1} , but short compared to ϵ^{-1} , yielding

$$\overline{\langle \delta R^2 \rangle}(t) = \ell^2 [1 + n_B(\omega_+) + n_B(\omega_-)] \cosh(\epsilon t). \quad (31)$$

The fluctuations of the center-of-mass are thus exponentially amplified due to the parametric modulation of the magnetic field. Notice that even at zero temperature, where initially the two bosonic modes are unoccupied [$n_B(\omega_{\pm}) = 0$ in Eq. (31)], parametric amplification takes place, triggered by the quantum fluctuations of the center-of-mass coordinate.

IV. EFFECTS OF DAMPING AND ANHARMONICITIES ON THE PARAMETRIC RESONANCE OF THE MAGNETOPLASMONS

In this section, we analyze the role played by dissipation and non-linearities on the parametric amplification of the magnetoplasmons. As we will show, these two effects are indeed limiting the exponential amplification of the bosonic modes presented in Sec. III. It is therefore relevant to study them as they will inevitably affect experiments.

A. Mean-field equations of motion

In what follows, we adopt a different strategy than the one used in Sec. III, and focus on the time evolution of the averaged quantities (25) and (26) as we are primarily interested in the fluctuations of the center-of-mass coordinate, Eq. (29). To this end, we use the time evolution of the reduced density matrix ρ associated to the center-of-mass degrees of freedom,

$$\dot{\rho} = -\frac{i}{\hbar} [H_{\text{cm}}, \rho] + \sum_{\sigma=\pm} \frac{\gamma_{\sigma}}{2} (2b_{\sigma}\rho b_{\sigma}^{\dagger} - b_{\sigma}^{\dagger}b_{\sigma}\rho - \rho b_{\sigma}^{\dagger}b_{\sigma}). \quad (32)$$

The first term on the right-hand side of Eq. (32) accounts for the unitary dynamics of the center-of-mass density matrix that evolves according to the Hamiltonian (12). The second term accounts for dissipative processes coming from the anharmonicities [and hence from the coupling Hamiltonian (9)], as well as stemming from other sources of dissipation, such as radiative damping, interaction with phonons, Joule heating due to the eddy currents generated by the electric field associated with the time-dependent vector potential (4), etc. In Eq. (32), we take dissipation phenomenologically into account by assuming the Lindblad form for the reduced density matrix.³⁰ We denote γ_+ and γ_- the damping rates for the two magnetoplasmon modes. Notice that, in general, $\gamma_+ \neq \gamma_-$ due to the difference in energy of the two modes, as recently demonstrated in the context of the Landau damping of the magnetoplasmon excitations in metallic nanoparticles.¹⁶

From the master equation (32), the time evolution of the quantum average of an operator \mathcal{O} is easily determined using the identity $\langle \dot{\mathcal{O}} \rangle(t) = \text{tr}\{\dot{\rho}(t)\mathcal{O}\}$. Introducing the notation $N = N_+ + N_-$, $\Delta N = N_+ - N_-$, $\gamma = (\gamma_+ + \gamma_-)/2$, and $\Delta\gamma = (\gamma_+ - \gamma_-)/2$, we find from Eqs. (12) and (32)

$$\begin{aligned} \dot{N} = & i\omega_c \delta f(t) (\varphi - \varphi^*) - \gamma N - \Delta\gamma \Delta N \\ & - i\alpha\omega_0 \left[2(\varphi^* - \varphi) + \langle b_+^\dagger b_+^\dagger b_-^\dagger b_-^\dagger \rangle + \langle b_+^\dagger b_+^\dagger b_-^\dagger b_- \rangle \right. \\ & + \langle b_+^\dagger b_-^\dagger b_-^\dagger b_- \rangle - \langle b_+^\dagger b_+ b_+ b_- \rangle - \langle b_-^\dagger b_+ b_- b_- \rangle \\ & \left. - \langle b_+ b_+ b_- b_- \rangle \right], \end{aligned} \quad (33a)$$

$$\Delta\dot{N} = -\gamma\Delta N - \Delta\gamma N, \quad (33b)$$

$$\begin{aligned} \dot{\varphi} = & -i[\omega_+ + \omega_- + \omega_c \delta f(t)] \varphi - i\frac{\omega_c}{2} \delta f(t) (N+1) \\ & - \gamma\varphi - i\frac{\alpha\omega_0}{2} \left(2 + 6\varphi + 2\varphi^* + 4N + \langle b_+^\dagger b_+^\dagger b_-^\dagger b_- \rangle \right. \\ & + \langle b_+^\dagger b_-^\dagger b_-^\dagger b_- \rangle + 4\langle b_+^\dagger b_-^\dagger b_+ b_- \rangle + \langle b_+^\dagger b_+^\dagger b_+ b_+ \rangle \\ & + \langle b_-^\dagger b_-^\dagger b_- b_- \rangle + 3\langle b_+^\dagger b_+ b_+ b_- \rangle + 3\langle b_-^\dagger b_+ b_- b_- \rangle \\ & \left. + 2\langle b_+ b_+ b_- b_- \rangle \right). \end{aligned} \quad (33c)$$

To first order in the small anharmonicity parameter $\alpha \ll 1$, the four-operator correlators appearing in Eq. (33) must be evaluated to order α^0 . Since for $\alpha = 0$, the Hamiltonian of Eq. (12) is quadratic in the bosonic operators, one can apply Wick's theorem, yielding the mean-field equations of motion

$$\begin{aligned} \dot{N} = & i\omega_c \delta f(t) (\varphi - \varphi^*) - \gamma N - \Delta\gamma \Delta N \\ & + 2i\alpha\omega_0 [(\varphi - \varphi^*) (N+1) + \varphi^2 - \varphi^{*2}], \end{aligned} \quad (34a)$$

$$\Delta\dot{N} = -\gamma\Delta N - \Delta\gamma N, \quad (34b)$$

$$\begin{aligned} \dot{\varphi} = & -i[\omega_+ + \omega_- + \omega_c \delta f(t)] \varphi - i\frac{\omega_c}{2} \delta f(t) (N+1) \\ & - \gamma\varphi - i\alpha\omega_0 \left[(N+1)^2 + 2\varphi^* \varphi + 2\varphi^2 \right. \\ & \left. + (3\varphi + \varphi^*) (N+1) \right]. \end{aligned} \quad (34c)$$

Introducing

$$\varphi(t) = \exp \left(-i \int_0^t ds [\omega_+ + \omega_- + \omega_c \delta f(s)] \right) \tilde{\varphi}(t), \quad (35)$$

to first order in $\eta \ll 1$ and within the rotating wave approximation close to resonance ($\Omega \simeq \omega_+ + \omega_-$), we find from Eq. (34)

$$\begin{aligned} \dot{N} = & \epsilon \left[e^{i(\Omega - \omega_+ - \omega_-)t} \tilde{\varphi} + \text{c.c.} \right] - \gamma N - \Delta\gamma \Delta N \\ & - 2\alpha\omega_0 \frac{\epsilon}{\Omega} \left[e^{i(\Omega - \omega_+ - \omega_-)t} \tilde{\varphi} + \text{c.c.} \right] (N+1), \end{aligned} \quad (36a)$$

$$\Delta\dot{N} = -\gamma\Delta N - \Delta\gamma N, \quad (36b)$$

$$\begin{aligned} \dot{\tilde{\varphi}} = & \frac{\epsilon}{2} e^{-i(\Omega - \omega_+ - \omega_-)t} (N+1) - \gamma\tilde{\varphi} - 3i\alpha\omega_0 \tilde{\varphi} (N+1) \\ & - \alpha\omega_0 \frac{\epsilon}{\Omega} \left\{ e^{-i(\Omega - \omega_+ - \omega_-)t} \left[(N+1)^2 + 2\tilde{\varphi}\tilde{\varphi}^* \right] \right. \\ & \left. - 2e^{i(\Omega - \omega_+ - \omega_-)t} \tilde{\varphi}^2 \right\}. \end{aligned} \quad (36c)$$

B. Effect of dissipative processes on the parametric resonance of the magnetoplasmon excitations

It is instructive to analyze the effect of damping alone on the parametric resonance of the magnetoplasmon excitations. To this end, we set the anharmonicity parameter α to zero in Eq. (36). Using the initial conditions $N(0) = n_B(\omega_+) + n_B(\omega_-)$, $\Delta N(0) = n_B(\omega_+) - n_B(\omega_-)$, and $\tilde{\varphi}(0) = 0$, we obtain the solution at resonance ($\Omega = \omega_+ + \omega_-$)

$$N(t) = \frac{e^{-\gamma t}}{\sqrt{\epsilon^2 + \Delta\gamma^2}} h'(t) - \frac{\epsilon^2}{\epsilon^2 + \Delta\gamma^2 - \gamma^2}, \quad (37a)$$

$$\begin{aligned} \Delta N(t) = & \frac{\epsilon e^{-\gamma t}}{\sqrt{\epsilon^2 + \Delta\gamma^2}} \left[\frac{\epsilon}{\sqrt{\epsilon^2 + \Delta\gamma^2}} \left(\Delta N(0) - \frac{\Delta\gamma}{\gamma} \right) \right. \\ & \left. - \frac{\Delta\gamma}{\epsilon} h(t) \right] + \frac{\Delta\gamma}{\gamma} \frac{\epsilon^2}{\epsilon^2 + \Delta\gamma^2 - \gamma^2}, \end{aligned} \quad (37b)$$

$$\begin{aligned} \tilde{\varphi}(t) = & \frac{\epsilon e^{-\gamma t}}{2\sqrt{\epsilon^2 + \Delta\gamma^2}} \left[\frac{\Delta\gamma}{\sqrt{\epsilon^2 + \Delta\gamma^2}} \left(\Delta N(0) - \frac{\Delta\gamma}{\gamma} \right) \right. \\ & \left. + h(t) \right] - \frac{\epsilon}{2\gamma} \frac{\gamma^2 - \Delta\gamma^2}{\epsilon^2 + \Delta\gamma^2 - \gamma^2}, \end{aligned} \quad (37c)$$

where we defined the function

$$\begin{aligned} h(t) = & \left[\frac{\epsilon^2}{\epsilon^2 + \Delta\gamma^2 - \gamma^2} + N(0) \right] \sinh \left(\sqrt{\epsilon^2 + \Delta\gamma^2} t \right) \\ & + \left[\frac{\gamma\epsilon^2}{\epsilon^2 + \Delta\gamma^2 - \gamma^2} - \Delta\gamma \Delta N(0) \right] \\ & \times \frac{\cosh \left(\sqrt{\epsilon^2 + \Delta\gamma^2} t \right)}{\sqrt{\epsilon^2 + \Delta\gamma^2}}, \end{aligned} \quad (38)$$

and $h'(t)$ denotes its derivative with respect to time.

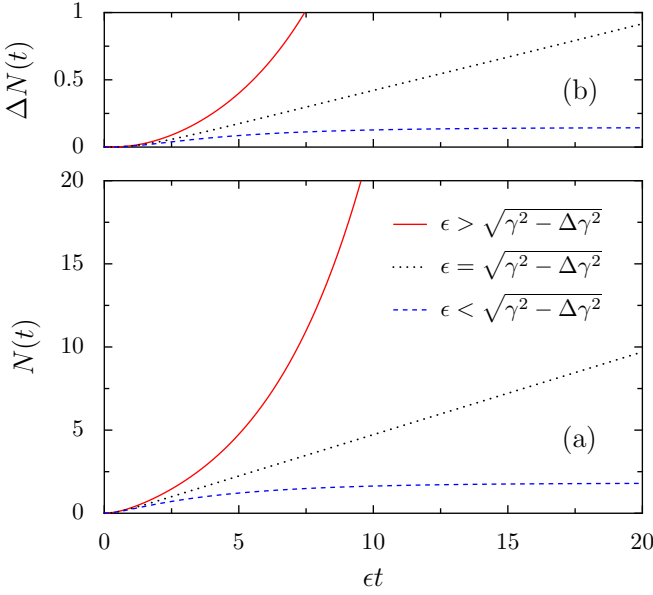


FIG. 2. (Color online) (a) Total number $N(t)$ and (b) difference $\Delta N(t)$ of bosonic excitations between the modes $+$ and $-$ as a function of time from Eqs. (37a) and (37b), respectively. In the figure, $T = 0$, $\Delta\gamma = -\epsilon/10$, and $\gamma = 3\epsilon/4$, $\sqrt{1.01}\epsilon$, and $5\epsilon/4$ for the solid (red), dotted (black), and dashed (blue) lines, respectively.

Thus, for $\epsilon > \sqrt{\gamma^2 - \Delta\gamma^2}$, the solutions (37) are exponentially amplified, as shown by the solid (red) lines in Fig. 2. Indeed, from Eq. (37), we obtain for $t \gg (\sqrt{\epsilon^2 + \Delta\gamma^2} - \gamma)^{-1}$ and $\epsilon > \sqrt{\gamma^2 - \Delta\gamma^2}$

$$N(t) \simeq \frac{e^{(\sqrt{\epsilon^2 + \Delta\gamma^2} - \gamma)t}}{2} \left[N(0) - \frac{\Delta\gamma}{\sqrt{\epsilon^2 + \Delta\gamma^2}} \Delta N(0) + \frac{\epsilon^2}{\sqrt{\epsilon^2 + \Delta\gamma^2} (\sqrt{\epsilon^2 + \Delta\gamma^2} - \gamma)} \right], \quad (39a)$$

$$\Delta N(t) \simeq -\frac{\Delta\gamma}{\sqrt{\epsilon^2 + \Delta\gamma^2}} N(t), \quad (39b)$$

$$\tilde{\varphi}(t) \simeq \frac{\epsilon}{2\sqrt{\epsilon^2 + \Delta\gamma^2}} N(t). \quad (39c)$$

On the contrary, for $\epsilon < \sqrt{\gamma^2 - \Delta\gamma^2}$, the system reaches the stationary solution

$$N^{\text{st}} = \frac{\epsilon^2}{\gamma^2 - \Delta\gamma^2 - \epsilon^2}, \quad (40a)$$

$$\Delta N^{\text{st}} = -\frac{\Delta\gamma}{\gamma} N^{\text{st}}, \quad (40b)$$

$$\tilde{\varphi}^{\text{st}} = \frac{\epsilon}{2\gamma} (N^{\text{st}} + 1), \quad (40c)$$

as exemplified by the dashed (blue) lines in Fig. 2. In terms of bosonic excitations in the two modes $+$ and $-$, this translates into

$$N_{\pm}^{\text{st}} = \frac{\gamma_{\mp}}{\gamma_{+} + \gamma_{-} - \gamma_{+}\gamma_{-} - \epsilon^2}, \quad (41)$$

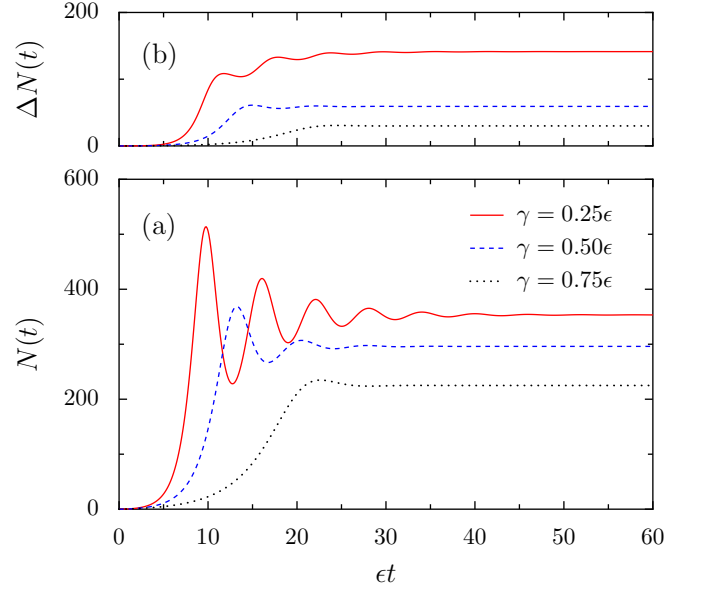


FIG. 3. (Color online) (a) Total number $N(t)$ and (b) difference $\Delta N(t)$ of bosonic excitations as a function of time from a numerical solution of Eq. (36) at resonance ($\Omega = \omega_{+} + \omega_{-}$) for various values of the damping constant γ . In the figure, $T = 0$, $\Delta\gamma = -\epsilon/10$, $\omega_0 = 10^3\epsilon$, $\Omega = \sqrt{5}\omega_0$ (corresponding to $\omega_c = \omega_0$), and $\alpha = 10^{-6}$.

i.e., the detailed balance $\gamma_{+} N_{+}^{\text{st}} = \gamma_{-} N_{-}^{\text{st}}$. For the special case $\epsilon = \sqrt{\gamma^2 - \Delta\gamma^2}$, we obtain from Eqs. (37) and (38), and for $\gamma t \gg 1$

$$N(t) \simeq \frac{1}{2} \left[\frac{\epsilon^2}{\gamma} \left(t - \frac{1}{2\gamma} \right) + N(0) - \frac{\Delta\gamma}{\gamma} \Delta N(0) \right], \quad (42a)$$

$$\Delta N(t) \simeq -\frac{\Delta\gamma}{2\gamma} \left[\frac{\epsilon^2}{\gamma} \left(t - \frac{3}{2\gamma} \right) + N(0) - \frac{\Delta\gamma}{\gamma} \Delta N(0) \right], \quad (42b)$$

$$\tilde{\varphi}(t) \simeq \frac{\epsilon}{4\gamma} \left[\frac{\epsilon^2}{\gamma} \left(t - \frac{3}{2\gamma} \right) + 2 + N(0) - \frac{\Delta\gamma}{\gamma} \Delta N(0) \right], \quad (42c)$$

i.e., a linear behavior of the solution as a function of t [see dotted (black) lines in Fig. 2].

C. Effect of anharmonicities on the parametric resonance of the magnetoplasmon excitations

We now turn to the full solution of Eq. (36), including anharmonic terms, i.e., $\alpha \neq 0$. To this end, we concentrate on the case where the pumping frequency is at resonance, i.e., $\Omega = \omega_{+} + \omega_{-}$. A numerical solution of Eq. (36) is presented in Fig. 3 for the case where $\epsilon > \sqrt{\gamma^2 - \Delta\gamma^2}$, i.e., when parametric amplification takes place in the absence of the anharmonic term ($\alpha = 0$), see Sec. IV B. For short times, both the total number of bosonic modes $N(t)$ [Fig. 3(a)] and the difference between the occupation of

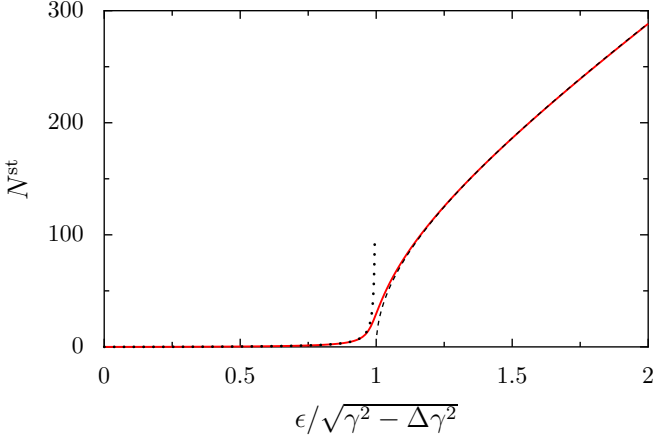


FIG. 4. (Color online) Stationary number of bosonic modes from Eq. (43) as a function of the driving frequency ϵ (solid red line). The asymptotic behaviors for $\epsilon < \sqrt{\gamma^2 - \Delta\gamma^2}$ [Eq. (40a)] and $\epsilon > \sqrt{\gamma^2 - \Delta\gamma^2}$ [Eq. (45)] are shown as a dotted and a dashed line, respectively. In the figure, $\alpha = 10^{-6}$ and $\omega_0/\gamma = 2 \times 10^3$.

the + and - modes, $\Delta N(t)$ [Fig. 3(b)] get exponentially amplified. This behavior is followed by a series of oscillations, to finally reach a stationary occupation due to the residual (anharmonic) interaction.

To better understand the numerical results of Fig. 3, we now search for the stationary solution of Eq. (36). Since we work to first order in the small parameters $\epsilon/\Omega \ll 1$ and $\alpha \ll 1$, we neglect the terms $\propto \alpha\epsilon/\Omega$ in Eq. (36), yielding

$$\left(1 - \frac{\Delta\gamma^2}{\gamma^2}\right) (3\alpha\omega_0)^2 N^{\text{st}} (N^{\text{st}} + 1)^2 - (\epsilon^2 + \Delta\gamma^2 - \gamma^2) N^{\text{st}} - \epsilon^2 = 0 \quad (43)$$

and

$$\Delta N^{\text{st}} = -\frac{\Delta\gamma}{\gamma} N^{\text{st}}, \quad (44a)$$

$$\tilde{\varphi}^{\text{st}} = \frac{\epsilon(N^{\text{st}} + 1)/2}{\gamma + 3i\alpha\omega_0(N^{\text{st}} + 1)}. \quad (44b)$$

The solution to Eq. (43) is shown in Fig. 4 as a red solid line as a function of the driving frequency ϵ . For $\epsilon > \sqrt{\gamma^2 - \Delta\gamma^2}$ and $\alpha = 0$, we know from our study in Sec. IV B that $N(t)$ diverges exponentially [cf. Eq. (39a)], such that we can safely assume for that case $N^{\text{st}} \gg 1$ in Eq. (43). If on top of that, $(\epsilon^2 + \Delta\gamma^2 - \gamma^2)N^{\text{st}} \gg \epsilon^2$, we obtain

$$N^{\text{st}} \simeq \frac{\sqrt{\epsilon^2 + \Delta\gamma^2 - \gamma^2}}{3\alpha\omega_0\sqrt{1 - \Delta\gamma^2/\gamma^2}}, \quad (45)$$

showing that the smaller α , the more efficient is the parametric amplification. The asymptotic behavior of Eq. (45) is shown as a dashed line in Fig. 4 and compares well with the full solution of Eq. (43) (solid red

line). For the case $\epsilon < \sqrt{\gamma^2 - \Delta\gamma^2}$, we find that parametric amplification does not take place, and that the stationary number of bosonic modes is, to first order in α , given by Eq. (40a), as shown in Fig. 4 by a dotted line.

V. EXPERIMENTAL CONSEQUENCES

Before presenting the various possible experimental consequences of our proposal, we start this section by discussing how one can trigger the parametric resonance of the magnetoplasmon collective modes in realistic quasi-two-dimensional semiconductor quantum dots. For a constant average electron density in the dot, the confinement frequency entering the single-particle potential (3) is given by^{18,31}

$$\omega_0 = \frac{e}{\sqrt{4\pi\epsilon_0\epsilon_r m^* r_s^3 N_e^{1/4}}} \quad (46)$$

and can be tuned by the gate voltage forming the quantum dot through the number N_e of electrons. In Eq. (46), ϵ_0 is the vacuum permittivity, ϵ_r the relative static dielectric constant of the considered material, r_s the Wigner-Seitz radius, and m^* the effective electron mass. Using the values for a GaAs quantum dot ($\epsilon_r = 13$, $m^* = 0.067m$ with m the bare electronic mass, $r_s = 1.5a_B^*$ with $a_B^* = 4\pi\epsilon_0\epsilon_r\hbar^2/m^*e^2$ the effective Bohr radius), and assuming $N_e = 10^3$, we have $\omega_0 = 1.7$ THz ($\hbar\omega_0 = 1.1$ meV). In a static magnetic field of $B_0 = 2$ T, the eigenfrequencies (14) are given by $\omega_+ = 5.7$ THz and $\omega_- = 0.5$ THz. The resonance condition $\Omega = \omega_+ + \omega_-$ for parametric amplification to occur [see Eq. (22)] thus requires in this case that the magnetic field is periodically modulated at the frequency $\Omega/2\pi = 0.99$ THz. The ongoing efforts towards the production of THz sources should allow one to attain such pumping frequencies in a near future.^{32–34} As we have shown in Secs. IV B and IV C, parametric amplification only occurs if the driving frequency ϵ of Eq. (19) is larger than the linewidth γ of the magnetoplasmons (assumed here to be the same for both collective modes). With a pumping strength $\eta = 0.1$, we obtain $\epsilon = 0.22$ THz. Assuming $\gamma \simeq \omega_0/10$ which is the typical value encountered in experiments,²⁰ parametric amplification should occur in this case at a rate given by $\epsilon - \gamma = 50$ GHz [see Eq. (39a)].

In what follows, we propose several ways of detecting the parametric amplification of the magnetoplasmons: First, these modes decay, among various processes, by radiative damping, emitting photons at the frequencies ω_+ and ω_- . Triggering the parametric resonance of the modes with the only help of a modulated magnetic field should thus result in the spontaneous emission of photons at these frequencies, *without* using an external light source, as usually employed in far infrared spectroscopy experiments.

Second, as we have shown in Sec. III, the average fluctuations of the electronic center-of-mass coordinate are

directly related to the total number of bosonic excitations N in the system, $\langle \delta R^2 \rangle \sim N$. As the latter is exponentially amplified up to its stationary value (45) due to the anharmonicity, this should result in an expansion of the whole electronic cloud forming the quantum dot which can be measured by scanning tunneling microscopy.³⁵

Another measurable quantity^{36,37} related to the fluctuations of the center-of-mass coordinate is the magnetization of the quantum dot. Indeed, the magnetization operator for the electronic system described by the Hamiltonian (2) reads

$$\mathbf{M} = -\frac{e}{2m^*} \sum_{i=1}^{N_e} \mathbf{r}_i \times [\mathbf{p}_i + e\mathbf{A}(\mathbf{r}_i, t)]. \quad (47)$$

Using Eq. (4) and introducing center-of-mass and relative coordinates as in Sec. II, the magnetization operator separates into $\mathbf{M} = \mathbf{M}_{\text{cm}} + \mathbf{M}_{\text{rel}}$, with

$$\mathbf{M}_{\text{cm}} = -\frac{e}{2m^*} \left(L_Z + \frac{N_e e B(t)}{2} R^2 \right) \mathbf{e}_z \quad (48)$$

the contribution from the center-of-mass dynamics, and

$$\mathbf{M}_{\text{rel}} = -\frac{e}{2m^*} \sum_i \left(l'_{z,i} + \frac{eB(t)}{2} r_i'^2 \right) \mathbf{e}_z \quad (49)$$

the one from the relative coordinates. Since the latter is induced by a bath of fermionic quasiparticles, the modulation of the magnetic field does not lead to their parametric amplification. In fact, the only result of the modulation is a small periodic variation of the relative magnetization that averages to zero due to the fast modulation. As a consequence, the average magnetization resulting from the relative coordinates is constant in time and equals its value without parametric modulation, $\langle \mathbf{M}_{\text{rel}} \rangle$.

In contrast, the magnetization (48) resulting from the center-of-mass dynamics can be conveniently rewritten in terms of the bosonic operators (10). Taking its expectation value, and averaging over fast oscillating terms, we find, to first order in the parametric modulation strength η ,

$$\overline{\langle \mathbf{M}_{\text{cm}} \rangle}(t) = -\mu_B^* \mathbf{e}_z \left\{ \Delta N(t) + \frac{\omega_c/2}{\sqrt{\omega_0^2 + \omega_c^2/4}} [N(t) + 1] \right\}, \quad (50)$$

with $\mu_B^* = e\hbar/2m^*$ the effective Bohr magneton ($\mu_B^* = 0.86 \text{ meV/T}$ for GaAs). As both the total number of bosonic excitations $N(t)$ and the difference $\Delta N(t)$ are exponentially amplified due to the parametric modulation to finally reach stationary values [see Eqs. (44a) and (45)], the resulting magnetization of the dot should dramatically increase when the parametric amplification of the magnetoplasmons takes place as compared to its equilibrium value.

VI. CONCLUSION

In conclusion, we have studied the possibility of parametrically amplifying bosonic collective modes in finite-size fermionic systems. Specifically, we have shown that the magnetoplasmons in quasi-two-dimensional semiconductor quantum dots can be parametrically amplified by modulating the magnetic field perpendicular to the nanostructure. Moreover, we have demonstrated that damping mechanisms and anharmonicities of the confinement lead to a saturation of the parametric resonance. We have further discussed the implementation of our proposal in realistic experimental samples and suggested measurements that should present clear signatures of the parametric amplification of the magnetoplasmons.

Our predictions could in principle also apply to the magnetoplasmon modes in metallic nanoparticles. However, as the damping of these modes is much stronger than in quantum dots,¹⁶ it may be much more difficult to trigger the parametric amplification in metallic nanoparticles.

ACKNOWLEDGMENTS

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Appendix A: Parametric resonance of a single harmonic oscillator: classical vs. quantum treatment

In this Appendix, we consider a single harmonic oscillator of mass m whose frequency $\omega(t) = \omega_0[1 + \eta \sin(\Omega t)]$ is periodically modulated in time at the frequency Ω . Denoting q and p the position and the momentum of the oscillator, its Hamiltonian reads

$$H_{\text{osc}}(t) = \frac{p^2}{2m} + \frac{m}{2} \omega^2(t) q^2. \quad (\text{A1})$$

In what follows, we assume that the strength η of the periodic modulation is much smaller than one, allowing for a perturbative treatment of the Hamiltonian (A1). For an arbitrary η , the Floquet formalism would be the appropriate one.³⁸ Before we tackle the quantum-mechanical treatment of the Hamiltonian (A1), we recall in the next section the classical case.

1. Classical treatment

In order to find the classical trajectories associated to the Hamiltonian (A1), as well as to stress the analogy

with the quantum mechanical treatment of the parametric resonance presented in the Sec. A2, it is convenient to introduce the new variables

$$z = \frac{1}{\sqrt{2i}} \left(\sqrt{m\omega_0} q + \frac{ip}{\sqrt{m\omega_0}} \right), \quad (\text{A2a})$$

$$\bar{z} = \frac{1}{\sqrt{2i}} \left(\sqrt{m\omega_0} q - \frac{ip}{\sqrt{m\omega_0}} \right). \quad (\text{A2b})$$

Since z and \bar{z} are canonically conjugated, i.e., the Poisson bracket $\{\bar{z}, z\} = 1$, the Hamilton equations of motion read $\dot{z} = -\partial H_{\text{osc}}/\partial \bar{z}$ and $\dot{\bar{z}} = \partial H_{\text{osc}}/\partial z$. With Eq. (A2), the Hamiltonian (A1) transforms into

$$H_{\text{osc}}(t) = i\omega_0 [1 + \eta \sin(\Omega t)] \bar{z} z + i\eta \frac{\omega_0}{2} \sin(\Omega t) (\bar{z}^2 + z^2), \quad (\text{A3})$$

yielding

$$\dot{z} = -i\omega_0 [1 + \eta \sin(\Omega t)] z - i\omega_0 \eta \sin(\Omega t) \bar{z}. \quad (\text{A4})$$

Introducing

$$z(t) = \exp \left(-i\omega_0 \int_0^t ds [1 + \eta \sin(\Omega s)] \right) \mathcal{Z}(t), \quad (\text{A5})$$

Eq. (A4) becomes, to first order in η ,

$$\dot{\mathcal{Z}} = -i\omega_0 \eta \sin(\Omega t) e^{2i\omega_0 t} \bar{\mathcal{Z}}. \quad (\text{A6})$$

Close to the resonance condition $\Omega \simeq 2\omega_0$, we keep only the dominant term $\propto e^{-i(\Omega-2\omega_0)t}$ such that Eq. (A6) decouples and yields

$$\ddot{\mathcal{Z}} + i(\Omega - 2\omega_0)\dot{\mathcal{Z}} + \left(\frac{\eta\omega_0}{2} \right)^2 \mathcal{Z} = 0. \quad (\text{A7})$$

With the initial conditions $\mathcal{Z}(0) = z(0)$ and $\dot{\mathcal{Z}}(0) = \eta\omega_0 \bar{\mathcal{Z}}(0)/2$, we finally obtain

$$z(t) = \frac{e^{-i\omega_0 t}}{\Omega_- - \Omega_+} \left[(\Omega_- e^{i\Omega_+ t} - \Omega_+ e^{i\Omega_- t}) z(0) + i \frac{\eta\omega_0}{2} (e^{i\Omega_+ t} - e^{i\Omega_- t}) z^*(0) \right], \quad (\text{A8})$$

with

$$\Omega_{\pm} = \omega_0 - \frac{\Omega}{2} \pm \frac{1}{2} \sqrt{(\Omega - 2\omega_0)^2 - (\eta\omega_0)^2}. \quad (\text{A9})$$

Thus, when $|\Omega - 2\omega_0| < \eta\omega_0$, the frequencies Ω_{\pm} become imaginary, which leads to the exponential amplification of the motion of the oscillator. Indeed, coming back to the original variables of the Hamiltonian (A1) and assuming $\Omega = 2\omega_0$, one obtains

$$q(t) = q(0) \cos(\omega_0 t) e^{\eta\omega_0 t/2} + \frac{p(0)}{m\omega_0} \sin(\omega_0 t) e^{-\eta\omega_0 t/2}, \quad (\text{A10a})$$

$$p(t) = p(0) \cos(\omega_0 t) e^{-\eta\omega_0 t/2} - m\omega_0 q(0) \sin(\omega_0 t) e^{\eta\omega_0 t/2}. \quad (\text{A10b})$$

Notice that the parametric amplification only takes place within the above approximate treatment if initially, $q(0) \neq 0$.

2. Quantum treatment

At the quantum level, the Hamiltonian (A1) is conveniently rewritten as

$$H_{\text{osc}}(t) = \hbar\omega_0 \left[1 + \eta \sin(\Omega t) \right] \left(b^\dagger b + \frac{1}{2} \right) + \eta \frac{\hbar\omega_0}{2} \sin(\Omega t) (b^{\dagger 2} + b^2) \quad (\text{A11})$$

in terms of the lowering and raising bosonic operators

$$b = \frac{1}{\sqrt{2}} \left(\frac{q}{\ell_{\text{osc}}} + \frac{ip\ell_{\text{osc}}}{\hbar} \right), \quad (\text{A12a})$$

$$b^\dagger = \frac{1}{\sqrt{2}} \left(\frac{q}{\ell_{\text{osc}}} - \frac{ip\ell_{\text{osc}}}{\hbar} \right), \quad (\text{A12b})$$

respectively. Here, $\ell_{\text{osc}} = \sqrt{\hbar/m\omega_0}$ denotes the oscillator length. The Heisenberg equation of motion for the operator b reads

$$\dot{b} = -i\omega_0 [1 + \eta \sin(\Omega t)] b - i\omega_0 \eta \sin(\Omega t) b^\dagger \quad (\text{A13})$$

and is similar to the classical equation of motion (A4) for the classical variable z . It can thus be solved in exactly the same way, yielding

$$b(t) = \frac{e^{-i\omega_0 t}}{\Omega_- - \Omega_+} \left[(\Omega_- e^{i\Omega_+ t} - \Omega_+ e^{i\Omega_- t}) b(0) + i \frac{\eta\omega_0}{2} (e^{i\Omega_+ t} - e^{i\Omega_- t}) b^\dagger(0) \right], \quad (\text{A14})$$

with the frequencies Ω_{\pm} given in Eq. (A9). Thus, at the quantum-mechanical level, the lowering and raising operators are exponentially amplified whenever the resonance condition $|\Omega - 2\omega_0| < \eta\omega_0$ is fulfilled.

Considering that the harmonic oscillator is in its ground state before the parametric amplification takes place ($t < 0$), and assuming $\Omega = 2\omega_0$, the average number of bosonic excitations

$$\langle b^\dagger b \rangle(t) = \frac{1}{2} [\cosh(\eta\omega_0 t) - 1] \quad (\text{A15})$$

exponentially increases for increasing time. During that process, the average position and momentum are both vanishing, $\langle q \rangle(t) = \langle p \rangle(t) = 0$, while the corresponding fluctuations are exponentially amplified,

$$\langle \delta q^2 \rangle(t) = \frac{\ell_{\text{osc}}^2}{2} [\cosh(\eta\omega_0 t) + \cos(2\omega_0 t) \sinh(\eta\omega_0 t)], \quad (\text{A16a})$$

$$\langle \delta p^2 \rangle(t) = \frac{\hbar^2}{2\ell_{\text{osc}}^2} [\cosh(\eta\omega_0 t) - \cos(2\omega_0 t) \sinh(\eta\omega_0 t)]. \quad (\text{A16b})$$

The quantum fluctuations of the oscillator in its ground state thus provide a seed for the parametric resonance in the quantum case.

Appendix B: Parametric resonance of the magnetoplasmons: classical treatment

Classically, the center of mass Hamiltonian (7) in the absence of the quartic correction ($A_4 = 0$) can be solved by a series of two successive canonical transformations. First, we introduce

$$\xi = \frac{1}{\sqrt{2}}(X + iY), \quad P_\xi = \frac{1}{\sqrt{2}}(P_X - iP_Y), \quad (\text{B1a})$$

$$\xi^* = \frac{1}{\sqrt{2}}(X - iY), \quad P_{\xi^*} = \frac{1}{\sqrt{2}}(P_X + iP_Y), \quad (\text{B1b})$$

with Poisson brackets $\{\xi, P_\xi\} = \{\xi^*, P_{\xi^*}\} = 1$, such that the Hamiltonian (7) transforms into

$$H_{\text{cm}}(t) = \frac{P_\xi P_{\xi^*}}{M} + M \left(\omega_0^2 + \frac{\omega^2(t)}{4} \right) \xi \xi^* + i \frac{\omega(t)}{2} (\xi P_\xi - \xi^* P_{\xi^*}). \quad (\text{B2})$$

Second, we introduce the canonically-conjugated variables

$$z_+ = \frac{1}{\sqrt{2i}} \left(\beta \xi^* + \frac{iP_\xi}{\beta} \right), \quad \bar{z}_+ = \frac{1}{\sqrt{2i}} \left(\beta \xi - \frac{iP_{\xi^*}}{\beta} \right), \quad (\text{B3a})$$

$$z_- = \frac{1}{\sqrt{2i}} \left(\beta \xi + \frac{iP_{\xi^*}}{\beta} \right), \quad \bar{z}_- = \frac{1}{\sqrt{2i}} \left(\beta \xi^* - \frac{iP_\xi}{\beta} \right), \quad (\text{B3b})$$

with $\beta = \sqrt{M(\omega_0^2 + \omega_c^4/4)^{1/2}}$ and the Poisson brackets $\{\bar{z}_\pm, z_\pm\} = 1$. In terms of these variables, the Hamiltonian (B2) reads

$$H_{\text{cm}}(t) = i \sum_{\sigma=\pm} \omega_\sigma [1 + \sigma \delta f(t)] \bar{z}_\sigma z_\sigma + i \frac{\omega_c}{2} \delta f(t) (\bar{z}_+ \bar{z}_- + z_+ z_-), \quad (\text{B4})$$

where the function $\delta f(t)$ is defined in Eq. (13) and the frequencies ω_\pm are given in Eq. (14). The Hamilton equations of motion associated to the Hamiltonian (B4) thus read

$$\dot{z}_\pm = -\frac{\partial H_{\text{cm}}}{\partial \bar{z}_\pm} = -i\omega_\pm [1 \pm \delta f(t)] z_\pm - i \frac{\omega_c}{2} \delta f(t) \bar{z}_\mp. \quad (\text{B5})$$

Being similar to the quantum mechanical Heisenberg equations of motion (15) for the bosonic operators b_\pm , the above equation can be solved in exactly the same way as in Sec. III. Going back to the original variables of the center of mass Hamiltonian (7), we find that close to resonance $|\Omega - \omega_+ - \omega_-| \ll \epsilon$ with ϵ given in Eq. (19), the classical motion of the center of mass is described by

the trajectories

$$X(t) = \{X(0) [\cos(\omega_+ t) + \cos(\omega_- t)] - Y(0) [\sin(\omega_+ t) - \sin(\omega_- t)]\} \frac{e^{\epsilon t/2}}{2} + \{P_X(0) [\sin(\omega_+ t) + \sin(\omega_- t)] + P_Y(0) [\cos(\omega_+ t) - \cos(\omega_- t)]\} \frac{e^{-\epsilon t/2}}{2\beta^2}, \quad (\text{B6a})$$

$$Y(t) = \{Y(0) [\cos(\omega_+ t) + \cos(\omega_- t)] + X(0) [\sin(\omega_+ t) - \sin(\omega_- t)]\} \frac{e^{\epsilon t/2}}{2} + \{P_Y(0) [\sin(\omega_+ t) + \sin(\omega_- t)] - P_X(0) [\cos(\omega_+ t) - \cos(\omega_- t)]\} \frac{e^{-\epsilon t/2}}{2\beta^2}. \quad (\text{B6b})$$

In Fig. 5, we show typical trajectories of the center of mass from Eq. (B6) in absence of parametric modulation [Fig. 5(a)] and with parametric modulation [Fig. 5(b)]. Without parametric modulation, the center of mass has trajectories corresponding to a two-dimensional harmonic oscillator in a perpendicular magnetic field [Fig. 5(a)]. The corresponding radial coordinate shown in Fig. 5(c) thus regularly oscillates between 0 and its maximal amplitude. In contrast, in the presence of parametric modulation [Figs. 5(b) and 5(d)], the motion of the center of mass is exponentially amplified.

Appendix C: Toy model: Parametric modulation of two coupled fermionic modes

In this Appendix, we consider two coupled fermionic modes with frequencies ϵ_+ and ϵ_- whose corresponding Hamiltonian is parametrically modulated. In the sequel, we show that due to the fermionic nature of the modes, Pauli principle prevents the parametric amplification to occur.

As a toy model, we consider a Hamiltonian similar to Eq. (12) without quartic correction, except that the bosonic operators in Eq. (12) are replaced by fermionic ones,

$$H_{\text{toy}}(t) = \sum_{\sigma=\pm} \hbar \epsilon_\sigma [1 + \sigma \eta \sin(\Omega t)] c_\sigma^\dagger c_\sigma + \eta \frac{\hbar \Delta}{2} \sin(\Omega t) (c_+^\dagger c_-^\dagger + c_- c_+). \quad (\text{C1})$$

Here, $c_\sigma^\dagger/c_\sigma$ creates/annihilates a fermion in mode $\sigma = \pm$. In Eq. (C1), $\eta \ll 1$ is the strength of the parametric modulation at frequency Ω and Δ is the frequency scale of a broken symmetry term. The corresponding Heisenberg equations of motion read

$$\dot{c}_\pm = -i\epsilon_\pm [1 \pm \eta \sin(\Omega t)] c_\pm \mp i\eta \frac{\Delta}{2} \sin(\Omega t) c_\mp^\dagger. \quad (\text{C2})$$

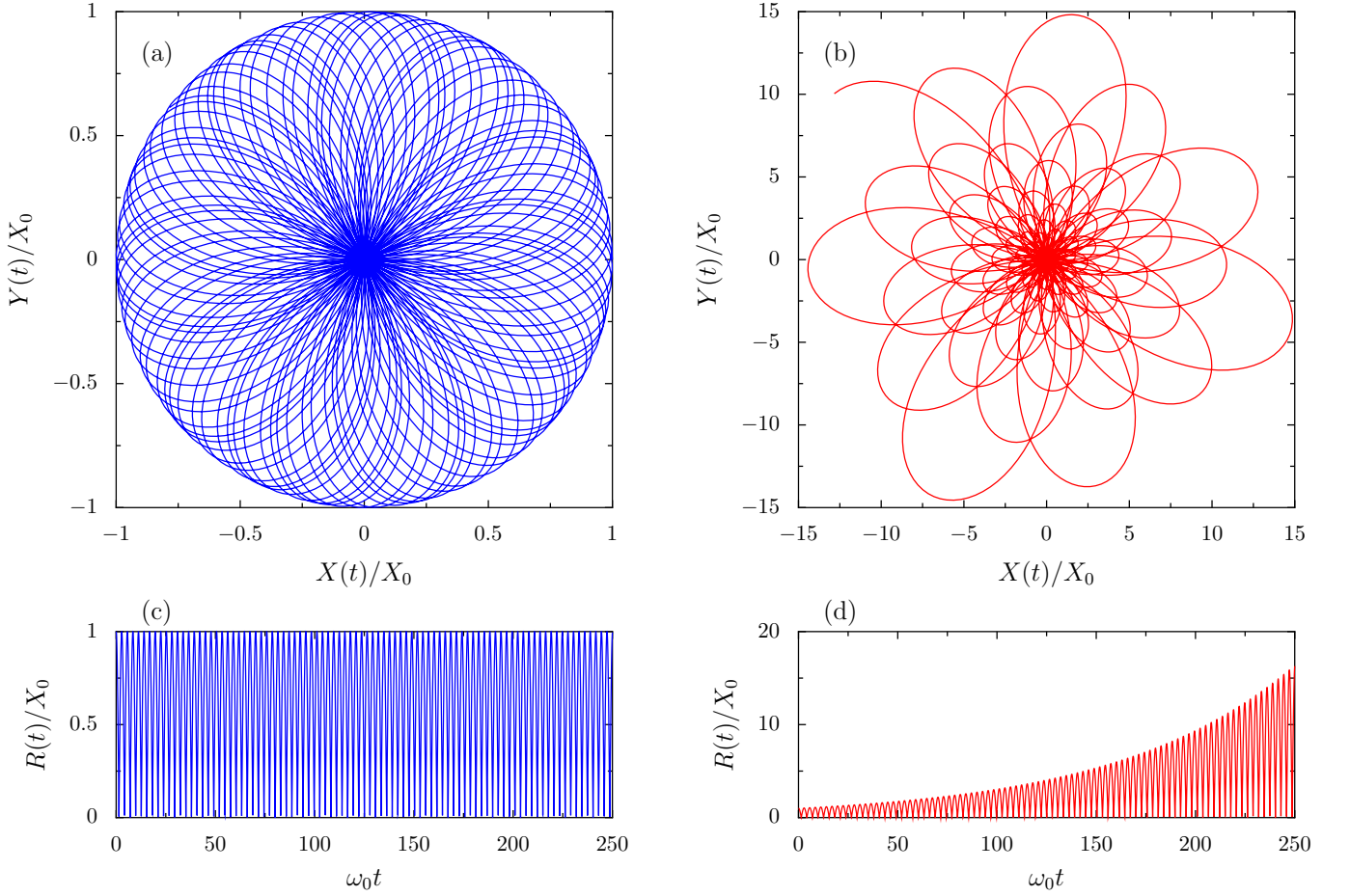


FIG. 5. (Color online) Upper panel: Typical trajectory $(X(t), Y(t))$ of the center of mass coordinate (a) in the absence of parametric modulation ($\eta = 0$) and (b) with parametric modulation ($\eta = 0.1$) for times up to $t_{\max} = 250/\omega_0$. Lower panel: radial coordinate $R(t) = \sqrt{X^2(t) + Y^2(t)}$ as a function of time for (c) $\eta = 0$ and (d) $\eta = 0.1$. For all trajectories, the initial conditions are $X(0) = X_0$, $Y(0) = 0$ and $P_X(0) = P_Y(0) = 0$. In the figure, the cyclotron frequency $\omega_c = \omega_0$ and the pumping frequency is at resonance, i.e., $\Omega = \omega_+ + \omega_-$.

Along the same lines as in Sec. III, introducing the new operators

$$c_{\pm}(t) = \exp\left(-i\epsilon_{\pm} \int_0^t ds [1 \pm \eta \sin(\Omega s)]\right) \tilde{c}_{\pm}(t), \quad (\text{C3})$$

we obtain for Eq. (C2) to first order in η and within the rotating wave approximation, which is valid for $\Omega \simeq \epsilon_+ + \epsilon_-$,

$$\ddot{\tilde{c}}_{\pm} + i(\Omega - \epsilon_+ - \epsilon_-) \dot{\tilde{c}}_{\pm} + \left(\frac{\eta\Delta}{4}\right)^2 \tilde{c}_{\pm} = 0. \quad (\text{C4})$$

Solving for the latter equation leads to

$$c_{\pm}(t) = \frac{e^{-i\epsilon_{\pm}t/\hbar}}{\Omega_- - \Omega_+} \left[(\Omega_- e^{i\Omega_+t} - \Omega_+ e^{i\Omega_-t}) c_{\pm}(0) \pm i \frac{\eta\Delta}{4} (e^{i\Omega_+t} - e^{i\Omega_-t}) c_{\mp}^{\dagger}(0) \right], \quad (\text{C5})$$

$$\Omega_{\pm} = \frac{\epsilon_+ + \epsilon_- - \Omega}{2} \pm \frac{1}{2} \sqrt{(\Omega - \epsilon_+ - \epsilon_-)^2 + \frac{(\eta\Delta)^2}{4}}. \quad (\text{C6})$$

Thus, contrarily to the bosonic case [cf. Eq. (22)], the frequencies Ω_{\pm} are real for any pumping frequency Ω , such that the fermionic operators (C5) are not parametrically amplified. This is a direct consequence of Pauli's principle.

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